XXXXXXXXXXXXXXXXXXXXXX

Keller, Joseph B.
Spherical, cylindrical and one-dimensional flows of compressible fluids.

IMM-NYU 185 September 1952

IMM-NYU 185
September 1952

SPHERICAL, CYLINDRICAL AND ONE-DIMENSIONAL FLOWS OF COMPRESSIBLE FLUIDS

by Joseph B. Keller

This report represents results obtained at the Institute for Mathematics and Mechanics, New York University, under the auspices of Contract Nonr-285(02).

Spherical, Cylindrical and One-Dimensional Flows of Compressible Fluids

Joseph B. Keller

I. Introduction

Relatively few boundary value problems involving spherical or cylindrical flows of compressible fluids have been solved exactly. In order to solve more problems of this type, particularly those involving variable entropy, we have investigated the flow differential equations, at first without regard to initial or boundary conditions. In this way a class of non-isentropic solutions of the differential equations, depending upon an arbitrary function, has been obtained. Then, in the second phase of the investigation, the arbitrary function is adjusted to satisfy particular initial or boundary conditions. In this way the free expansion of a sphere of gas into a vacuum has been treated, as well as the propagation of finite and strong shocks in variable media. The latter treatment includes Primakoff's point-blast solution as a special case.

The method of procedure is quite simple. In section II the problem is formulated in Lagrangian variables. In section III a class of solutions is obtained by the method of separation of variables. In section IV these solutions are specialized to isentropic flow. In section V the solutions are applied to the determination of strong shock waves propagating in variable media. In section VI finite shocks in variable media are considered.

II. Formulation

We consider the motion of an inviscid, non-heat-conducting fluid, obeying the polytropic equation of state. The one, two and three dimensional cases will be treated together. In the three dimensional case $y(h,t)$ represents

the radius at time t of the particle with the Lagrangian coordinate h, which is defined by the equation

$$(1) \qquad h = \int_{y(0,t)}^{y(h,t)} r^{n-1}\rho(r,t)\, dr \qquad n = 1,2,3.$$

In the above equation $\rho(r,t)$ is the density at time t and radius r, and n is the dimension, which is 3 in the spherical case. In the two dimensional case y represents radial distance from an axis, and in one dimension y is a cartesian coordinate.

It is further assumed that all flow variables depend upon y and t only and that flow occurs only in the y direction. This is the assumption of spherical, cylindrical or planar symmetry.

From the definition of y, the particle velocity u is given by

$$(2) \qquad u = y_t \quad .$$

Similarly from (1) the density ρ or specific volume τ are given by

$$(3) \qquad \tau = \rho^{-1} = y^{n-1} y_h \quad .$$

Because of our assumptions concerning inviscidity and non-conduction, the entropy s of a particle is independent of time (at least between successive shocks). Thus we have

$$(4) \qquad s = s(h) \quad .$$

The function s(h) is given by initial data or by shock conditions, and is assumed to be known.

The pressure p is given by the equation of state

$$(5) \qquad p = p(\rho, s) = g(\tau, s) \quad .$$

For a polytropic gas or liquid

(6) $$g(\tau, s) = g_0 + A(s)\tau^{-\gamma} \quad .$$

The function $A(s)$, the adiabatic exponent γ, and the internal pressure g_0 are assumed to be known.

In terms of the above defined quantities, the equation of motion is

(7) $$y_{tt} = -y^{n-1}[g_\tau(y^{n-1}y_h)_h + g_s s_h] \quad .$$

For a polytropic gas or liquid this becomes, using (6),

(8) $$y_{tt} = \gamma A(s)(y^{n-1}y_h)^{-\gamma-1}(y^{n-1}y_h)_h\, y^{n-1} - A_h(y^{n-1}y_h)^{-\gamma}\, y^{n-1} \quad .$$

Equation (8) is a second order partial differential equation for $y(h,t)$. The coefficient A is assumed to be a known function of s, and by (4), of h.

The problem we consider is that of finding solutions of (8). From a particular solution, the flow variables can be found by using (2), (3), (4) and (5).

III. Product Solutions

Let us seek product solutions of (8) of the form

(9) $$y(h,t) = f(h)j(t) \quad .$$

Inserting (9) into (8), and separating variables we obtain

(10) $$j'' - \lambda j^{n(1-\gamma)-1} = 0$$

(11) $$-A[(f^{n-1}f')^{-\gamma}]'f^{n-2} - A'(f^{n-1}f')^{-\gamma}\, f^{n-2} = \lambda \quad .$$

In these equations λ is an arbitrary separation parameter and primes represent derivatives with respect to t in (10) and with respect to h in (11).

To solve (10) we multiply by j' and integrate, obtaining

(12) $$\frac{1}{2}(j')^2 - \frac{\lambda}{n(1-\gamma)}\, j^{n(1-\gamma)} = \frac{1}{2}a \qquad (\gamma \neq 1)$$

$$\frac{1}{2}(j')^2 - \lambda \log j = \frac{1}{2}a \qquad (\gamma = 1) \quad . \tag{13}$$

Here a is an integration constant. Thus, unless j is constant, which is possible only if $\lambda = 0$ or $j = 0$, we have the solution

$$\int^j \left[\frac{2\lambda}{n(1-\gamma)}\, j^{n(1-\gamma)} + a\right]^{-1/2} dj = t \qquad (\gamma \neq 1) \tag{14}$$

$$\int^j [2\lambda \log j + a]^{-1/2}\, dj = t \qquad (\gamma = 1) \quad . \tag{15}$$

The integration constant in (14), (15) has been set equal to zero without loss of generality since it merely shifts the origin of time.

To solve (11) we differentiate out and obtain

$$\gamma A[f'' f^{n-1} + (n-1) f^{n-2} (f')^2](f^{n-1} f')^{-\gamma-1} \tag{16}$$
$$- f^{n-2}(f^{n-1} f')^{-\gamma} A' = \lambda \quad .$$

We now consider the inverse function $h = h(f)$ and denote $h'(f)$ by $q(f)$. Then (16) becomes

$$\gamma A[-q' q^{-3} f^{n-1} + (n-1) f^{n-2} q^{-2}][f^{n-1} q^{-1}]^{-\gamma-1} f^{n-2} \tag{17}$$
$$+ f^{n-2}[f^{n-1} q^{-1}]^{-\gamma} A'(h) = \lambda \quad .$$

We shall first treat the case $\gamma \neq 1$ by introducing $z(f)$ and $B(f)$ defined by

$$z = q^{\gamma-1} \quad , \qquad B(f) = A[h(f)] \quad . \tag{18}$$

With these definitions, (17) becomes (prime denoting differentiation with respect to f)

$$z' + z\{-(n-1)(\gamma-1) f^{-1} + \frac{\gamma-1}{\gamma} (\log B)'\} \tag{19}$$
$$+ \frac{\lambda(\gamma-1)}{\gamma^3} f^{(n-1)(\gamma-1)+1} = 0 \quad .$$

An integrating factor of (19) is $f^{-(n-1)(\gamma-1)}B^{\frac{\gamma-1}{\gamma}}$ and the solution is (with G a constant)

$$(20) \qquad z = f^{(n-1)(\gamma-1)}B^{\frac{1-\gamma}{\gamma}}\left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\,df\right] \quad .$$

Thus we have q from (18) and (20), and finally since $h' = q$,

$$(21) \qquad h = \int^{f} f^{n-1}B^{-1/\gamma}\left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\,df\right]^{\frac{\gamma}{\gamma-1}} df + E \quad .$$

Here E is an integration constant.

Equation (21) gives $f(h)$ implicitly, and thus provides a solution of (11) for $\gamma \neq 1$. The solution may be written more simply by defining $F(f)$ by

$$(22) \qquad F(f) = \left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\,df\right]^{\frac{\gamma}{\gamma-1}} \quad .$$

If $\lambda \neq 0$ this can be solved for $B(f)$ and yields

$$(23) \qquad B(f) = (-\lambda f)^{\gamma}(f')^{-\gamma}\,F \quad .$$

The solution for the flow variables can now be computed from (2)-(5), (9), (21), (22). We have for $\gamma \neq 1$ and $\lambda \neq 0$, remembering that $f = yj^{-1}$ from (9),

$$(24) \qquad u(y,t) = yj'j^{-1}$$

$$(25) \qquad \tau(y,t) = -\lambda y j^{n-1}/F'(yj^{-1})$$

$$(26) \qquad p(y,t) = g_0 + j^{-n\gamma}\,F(yj^{-1}) \quad .$$

Equations (24)-(26) give the flow quantities. In fact, these expressions yield a solution of the Eulerian equations of motion for an arbitrary function F provided that $j(t)$ is given by (14). This is the first main result of this paper. It is to be noted that in these solutions, u is proportional to y.

In the excluded case $\gamma \neq 1$, $\lambda = 0$ we have instead of (24)-(26) from (2)-(5), (9), (21)

$$(27) \qquad u(y,t) = yj'j^{-1} \qquad\qquad = yt^{-1}$$

$$(28) \qquad \tau(y,t) = j^n B^{1/\gamma}(yj^{-1}) G^{\frac{1}{1-\gamma}} = t^n b(yt^{-1})$$

$$(29) \qquad p(y,t) = g_o + j^{-n\gamma} G^{\frac{\gamma}{1-\gamma}} \quad = g_o + \ell t^{-n\gamma} \quad .$$

The new arbitrary function b and arbitrary constant ℓ have been introduced and the expressions simplified by using (14), which gives

$$(30) \qquad j(t) = a^{1/2} t \quad .$$

The corresponding solutions when $\gamma = 1$ are obtained from (17) by introducing $B(f)$ as in (18), after which (17) becomes

$$(31) \qquad q'q^{-1} - (n-1)f^{-1} + \frac{B'}{B} + \frac{\lambda f}{B} = 0 \quad .$$

The solution of (31) is

$$(32) \qquad q(f) = f^{n-1} B^{-1}(f) \exp \int^f -\lambda f B^{-1}(f)\,df \quad .$$

Thus

$$(33) \qquad h(f) = \int^f \left[f^{n-1} B^{-1}(f) \exp \int^f -\lambda f B^{-1}(f)\,df \right] df + E \quad .$$

Equation (33) yields the solution implicitly. Again we define

$$(34) \qquad F(f) = \exp \int^f -\lambda f B^{-1}(f)\,df \quad .$$

Then if $\lambda \neq 0$ we have

$$(35) \qquad B(f) = -\lambda f F (F')^{-1} \quad .$$

The solution for the flow variables, if $\gamma = 1$ and $\lambda \neq 0$, is again given by (24)-(26) with $\gamma = 1$ and $j(t)$ given by (15). Similarly for $\gamma = 1$ and $\lambda = 0$ the solution is given by (27)-(29) with $\gamma = 1$, $G = 1$ and $j(t)$ given by (30).

Equations (24)-(26) and (27)-(29) represent non-isentropic solutions of the flow equations depending upon an arbitrary function. In order to construct these solutions explicitly one need merely evaluate the integrals in (14) or (15). Some special cases of these integrals yield results which are listed below for later use.

$$(36)\quad j(t) = \left\{ \left[\frac{2\lambda}{n(1-\gamma)}\right]^{1/2} \left(\frac{n(\gamma-1)+2}{2}\right) t \right\}^{\frac{2}{n(\gamma-1)+2}}$$

$$\text{if } \gamma \neq 1,\ a = 0,\ \frac{n(\gamma-1)}{2} \neq -1$$

$$(37)\quad j(t) = e^{\pm\lambda^{1/2} t} \qquad \text{if } \gamma \neq 1,\ a = 0,\ \frac{n(\gamma-1)}{2} = -1$$

$$(38)\quad j(t) = \frac{\lambda}{2} t^2 + \alpha t + \beta \qquad \text{if } n(1-\gamma) = 1,\ \alpha,\ \beta \text{ arbitrary.}$$

IV. <u>Isentropic Case</u>

The above solution can be specialized to the isentropic case by setting A = B = constant. Then from (22) and (34) we obtain

$$(39)\quad F(f) = \left[G - \frac{\lambda(\gamma-1)}{2\gamma B^{1/\gamma}} f^2\right]^{\frac{\gamma}{\gamma-1}} \qquad \gamma \neq 1$$

$$(40)\quad F(f) = \exp \frac{-\lambda f^2}{2B} \qquad \gamma = 1 \quad .$$

Thus the solutions given by (24)-(26) become, for $\gamma \neq 1$ and $\lambda = 0$

$$(41)\quad u(y,t) = y j' j^{-1}$$

$$(42)\quad \tau(y,t) = j^n B^{1/\gamma} \left[G - \frac{\lambda(\gamma-1)}{2\gamma B^{1/\gamma}} y^2 j^{-2}\right]^{\frac{-1}{\gamma-1}}$$

$$(43)\quad p(y,t) = g_o + j^{-n\gamma} \left[G - \frac{\lambda(\gamma-1)}{2\gamma B^{1/\gamma}} y^2 j^{-2}\right]^{\frac{\gamma}{\gamma-1}} \quad .$$

In these solutions B and G are constants and j is given by (14). For $\lambda = 0$, and all γ the solution (27)-(29) applies with b constant.

For $\gamma = 1$ and all λ, (41) is unchanged, but the other equations become

(44) $\tau(y,t) = j^n B \exp \frac{-\lambda j^{-2} y^2}{2B}$

(45) $p(y,t) = g_o + j^{-n} \exp \frac{\lambda j^{-2} y^2}{2B}$.

Here B is a constant and j is given by (15).

As an example of these isentropic solutions, let j(t) be given by (36) and assume G = 0 in (42), (43). Then (41)-(43) yield the power solutions (if $\gamma \neq 1$, $\frac{n(\gamma-1)}{2} \neq -1$)

(46) $u(y,t) = \frac{2}{n(\gamma-1)+2}\, y t^{-1}$

(47) $\tau(y,t) = \left[\frac{\gamma B}{n}\left(\frac{n(\gamma-1)+2}{\gamma-1}\right)^2\right]^{\frac{1}{\gamma-1}} (yt^{-1})^{\frac{-2}{\gamma-1}}$

(48) $p(y,t) = g_o + B\left[\frac{\gamma B}{n}\left(\frac{n(\gamma-1)+2}{\gamma-1}\right)^2\right]^{\frac{\gamma}{1-\gamma}} (yt^{-1})^{\frac{2\gamma}{\gamma-1}}$.

V. Application: Strong Shocks In Variable Media

Let us suppose that a shock given by the equation $y = R(t)$ moves into a variable medium of density $\rho_o(y)$, pressure $p_o(y)$ and velocity zero. The pressure p, velocity u and density ρ just behind the shock are related to the corresponding quantities in front of it by the shock conditions. These conditions are

(49) $\frac{\rho}{\rho_o} = \frac{(\gamma+1)p + (\gamma-1)p_o}{(\gamma-1)p + (\gamma+1)p_o} \approx \frac{\gamma+1}{\gamma-1}$ $\qquad (\gamma \neq \pm 1)$

(50) $u = \frac{2(p - p_o)}{\{2\rho_o[(\gamma+1)p + (\gamma-1)p_o]\}^{1/2}} \approx \left[\frac{2p}{(\gamma+1)\rho_o}\right]^{1/2}$

(51) $\dot{R} = \left[\frac{(\gamma+1)p + (\gamma-1)p_o}{2\rho_o}\right]^{1/2} \approx \left[\frac{(\gamma+1)p}{2\rho_o}\right]^{1/2}$.

The second expressions on the right apply to a strong shock, for which $p \gg p_0$.

We shall now assume that the flow behind the shock is a "product" solution given by (24)-(26), and that the shock is strong. We wish to determine the functions $F(yj^{-1})$, $R(t)$, $\rho_0(y)$ and $p_0(y)$ for which such a solution is possible. To this end, we insert (24)-(26) into (49)-(51) and obtain

$$(52) \qquad \frac{F'(Rj^{-1})}{-\lambda R j^{n-1} \rho_0(R)} = \frac{\gamma+1}{\gamma-1}$$

$$(53) \qquad \frac{Rj'}{j} = \left[\frac{2g_0 + 2j^{-n\gamma} F(Rj^{-1})}{(\gamma+1)\rho_0(R)} \right]^{1/2}$$

$$(54) \qquad \dot{R} = \left[\frac{(\gamma+1)\{g_0 + j^{-n\gamma} F(Rj^{-1})\}}{2\rho_0(R)} \right]^{1/2} .$$

From (53) and (54) we have

$$(55) \qquad \frac{\dot{R}}{R} = \frac{(\gamma+1)}{2} \frac{j'}{j} .$$

Thus

$$(56) \qquad R(t) = R_0 [j(t)]^{\frac{\gamma+1}{2}} .$$

Inserting (56) into (53) and making use of (12), satisfied by $j(t)$, we find

$$(57) \qquad F(x) = \frac{\gamma+1}{2} \rho_0 \left(x^{\frac{\gamma+1}{\gamma-1}} R_0^{\frac{2}{1-\gamma}} \right) \cdot \left[\frac{2\lambda}{n(1-\gamma)} (xR_0^{-1})^{-2n} + D \right] R_0^{\frac{-2n\gamma}{\gamma-1}} x^{\frac{(2n+2)\gamma-2}{\gamma-1}} - g_0 (xR_0^{-1})^{\frac{2n\gamma}{\gamma-1}} .$$

Thus R is determined, F is related to ρ_0, and (53), (54) are satisfied, although R_0, D, and λ are still arbitrary. We now insert (56), (57) into (52) to determine F. We obtain, if $g_0 = 0$,

$$(58)\qquad F(x) = F_o x^n \left[\frac{2\lambda R_o^{2n}}{n(1-\gamma)} + Dx^{2n}\right]^{-1/2} .$$

From (57) and (58) we then find

$$(59)\qquad \rho_o(R) = \frac{2F_o R_o^{\frac{2n\gamma}{\gamma-1}}}{\gamma+1} \left[\frac{2\lambda R_o^{2n}}{n(1-\gamma)} + DR_o^{\frac{4n}{\gamma+1}} R^{\frac{2n(\gamma-1)}{\gamma+1}}\right]^{-3/2}$$

$$\cdot R^{\frac{(n-2)\gamma-3n+2}{\gamma+1}} R_o^{\frac{2(n-2)\gamma-6n+4}{(\gamma+1)(\gamma-1)}} .$$

In (58) and (59) F_o is an arbitrary constant. The corresponding results with $g_o \neq 0$ are somewhat more complicated.

We have thus obtained a solution with a strong shock moving into a variable medium at rest, with density given by (59). The constants F_o, R_o, λ and D are arbitrary in this equation, but only two essential combinations of these constants occur. The shock curve is given by (56), and $j(t)$ by (14). The flow is given by (24)-(26) with $F(x)$ given by (58). The flow might be produced by a piston following one of the particle paths

$$(60)\qquad Y(t) = Y_o j(t) .$$

The solution was deduced for $\gamma \neq \pm 1$ and $\lambda \neq 0$.

As an example of these solutions, let us suppose that $\rho_o(R)$ = constant. From (59) we find that this requires

$$(61)\qquad D = 0 \quad , \quad \gamma = \frac{3n-2}{n-2} .$$

The last condition can be fulfilled only for $n = 3$, in which case $\gamma = 7$. ($n = 1$ leads to $\gamma = -1$, which was excluded in the derivation, and $n = 2$ yields no value of γ.) In this case we have from (14), (58)

$$(62)\qquad j = \left[\frac{5\sqrt{-\lambda}}{3}\, t\right]^{1/10} , \qquad F(x) = x^3\, \frac{3F_o R_o^{-n}}{\sqrt{-\lambda}} .$$

The solution computed by using (62) in (24)-(26) (with $\lambda < 0$) is exactly the point blast solution of Taylor's type first found by H. Primakoff, and applicable to high energy explosions in water. [see [1], p. 424]

VI. Application: Finite Shocks In Variable Media

We may apply the above method to determine finite shocks in variable media. Then we must satisfy the exact shock conditions (49)-(51) rather than the strong shock conditions. Inserting (24)-(26) into (49)-(51) yields with $g_o = 0$

$$(63)\qquad \frac{F'(x)}{-\lambda R j^{n-1}\rho_o} = \frac{(\gamma+1)j^{-n\gamma}F + (\gamma-1)p_o}{(\gamma+1)p_o + (\gamma-1)j^{-n\gamma}F}\quad , \qquad x = Rj^{-1}$$

$$(64)\qquad Rj'j^{-1} = \frac{j^{-n\gamma}F - p_o}{\sqrt{\frac{\rho_o}{2}[(\gamma+1)j^{-n\gamma}F + (\gamma-1)p_o]}^{1/2}}$$

$$(65)\qquad \dot{R} = \frac{1}{\sqrt{2\rho_o}}\,[(\gamma+1)j^{-n\gamma}F + (\gamma-1)p_o]^{1/2}\quad .$$

Equations (63)-(65) are a set of two first order ordinary differential equations and one algebraic equation for the determination of the four functions $F(x)$, $R(t)$, $p_o(R)$ and $\rho_o(R)$. Equation (14) gives $j(t)$. This system is evidently underdetermined and will have infinitely many solutions. In the strong shock case, however, p_o did not occur and thus the system was determined.

To solve the above system we could impose some relation between p_o and ρ_o and eliminate both these functions by means of that relation and (64). A pair of simultaneous first order equations for $F(x)$ and $R(t)$ would result. However these equations can be treated separately since, from (64) and (65) we have by eliminating F,

$$(66)\quad \dot{R} = \frac{\gamma+1}{4} Rj'j^{-1} \pm \left[\frac{(\gamma+1)^2}{16}(Rj'j^{-1})^2 + \gamma p_o \rho_o^{-1}\right]^{1/2} .$$

This is an equation for R(t) if the ratio $p_o \rho_o^{-1}$ is given as a function of R. (Equation (66) also holds when $g_o \neq 0$.) After solving this, (63) can be solved for F, and then p_o and ρ_o can be obtained.

We will now investigate the shock curve R(t) given by (66), in the special case in which j(t) is given by (36) with C = 0 and $\gamma p_o \rho_o^{-1} = c_o^2$ is constant. The latter assumption means the temperature ahead of the shock is constant. Making use of (36), (66) becomes

$$(67)\quad \dot{R} = \frac{\gamma+1}{2[n(\gamma-1)+2]} Rt^{-1} \pm \left[\frac{(\gamma+1)^2}{4[n(\gamma-1)+2]^2} (Rt^{-1})^2 + c_o^2\right]^{1/2} .$$

If we now introduce $U(t) = Rt^{-1}$ in (67) we obtain an equation for U in which the variables separate. One solution is the constant solution

$$(68)\qquad U = \pm c_o\left(1 - \frac{\gamma+1}{n(\gamma-1)+2}\right)^{-1/2} .$$

In this case, which is physically possible only when $\frac{\gamma+1}{n(\gamma-1)+2} < 1$, the shock curve is the straight line

$$(69)\qquad R = Ut = \pm c_o\left(1 - \frac{\gamma+1}{n(\gamma-1)+2}\right)^{-1/2} t .$$

When U does not have the value given by (68), the equation for U yields

$$(70)\quad \int_{U_o}^{U(t)} \left\{\left(\frac{\gamma+1}{2n(\gamma-1)+4} - 1\right)U \pm \left[\frac{(\gamma+1)^2}{[2n(\gamma-1)+4]^2} U^2 + c_o^2\right]^{1/2}\right\}^{-1} dU = \log \frac{t}{t_o} .$$

Integrating (70) yields

$$(71)\quad bt = \left| \pm a + (a-1)\cos\theta \right|^{\frac{1-a}{2a-1}} \left(\sin\frac{\theta}{2}\right)^{\frac{-1}{a\pm a-1}} \left(\cos\frac{\theta}{2}\right)^{\frac{1}{\pm a-a+1}} .$$

Here $\theta = \cot^{-1}\frac{aU}{c_o}$, $a = \frac{\gamma+1}{2n(\gamma-1)+4}$ and b is a constant. Thus we have

$$(72)\qquad R = Ut = \frac{c_c}{ab}\cdot bt\,\cot\theta \quad .$$

Since bt is given in terms of θ by (71), R(t) is given parametrically in terms of θ. A graph of $\frac{ab}{c_o}R$ versus bt is given in Figure 1 for $n = 3$ and $\gamma = 1.4$, which represents a spherical shock in air. When the minus sign is chosen in (67)-(71) the same curves are obtained as with the plus sign, provided t is replaced by minus t.

For small values of t we have

$$(73)\qquad R \approx Kt^{2a}$$

where K is a constant. For $\gamma = 7$ and $n = 3$ we find $a = \frac{1}{5}$ and thus $R \sim t^{2/5}$, which is the behavior of Primakoff's point blast solution for small t. For large t, R behaves linearly in t, as in (69).

To determine p_o and F, we have from (65) and the condition $\gamma p_o \rho_o^{-1} = c_o^2$,

$$(74)\qquad F = \frac{2\rho_o\dot{R}^2 - (\gamma-1)p_o}{(\gamma+1)j^{-n\gamma}} = \frac{2\gamma p_o}{(\gamma+1)c_o^2}\dot{R}^2 j^{n\gamma} - \frac{\gamma-1}{\gamma+1}p_o j^{n\gamma} \quad .$$

Thus, if dot denotes time derivative and prime denotes derivative of a function with respect to its argument, we have

$$(75)\qquad \dot{F} = \frac{2\gamma}{(\gamma+1)c_o^2}(p_o'\dot{R}^3 j^{n\gamma} + 2p_o\dot{R}\ddot{R}j^{n\gamma} + n\gamma p_o\dot{R}^2 j^{n\gamma-1}j')$$

$$- \frac{\gamma-1}{\gamma+1}(p_o'\dot{R}j^{n\gamma} + n\gamma p_o j^{n\gamma-1}j')$$

$$= F'(\dot{R}j^{-1} - Rj^{-2}j') \quad .$$

Using (74), (75) in (63) yields the following equation for p_o

$$(76)\quad \left[\frac{2\gamma}{(\gamma+1)c_o^2}\left(p_o'\dot{R}^3 j^{n\gamma} + 2p_o\dot{R}\ddot{R}j^{n\gamma} + n\gamma p_o\dot{R}^2 j^{n\gamma-1}j'\right)\right.$$

$$\left.- \frac{\gamma-1}{\gamma+1}\left(p_o'\dot{R}j^{n\gamma} + n\gamma p_o j^{n\gamma-1}j'\right)\right]\left[-\lambda R j^{n-1}\gamma p_o c_o^{-2}\left(\dot{R}j^{-1} - Rj^2 j'\right)\right]^{-1}$$

$$= 2\gamma c_o^{-2}\dot{R}^2\left[\gamma+1 + \frac{(\gamma-1)2\gamma\dot{R}^2}{(\gamma+1)c_o^2} - \frac{(\gamma-1)^2}{\gamma+1}\right]^{-1} .$$

Upon introducing $\dot{p}_o = p_o'\dot{R}$, the above equation can be integrated yielding $p_o[R(t)]$ as a function of t. To obtain $p_o(R)$, we must insert $t = t(R)$ from (71), (72) into this result. Then when $p_o[R(t)]$ is known, F can be found from (74). These calculations will not be carried out here.

VII. Free Expansion and Other Problems

In this section some flows described by the preceding "product" solutions will be discussed. First, let us consider the flow given by (41)-(43) with $g_o = 0$. This describes the isentropic flow of a gas. If $j'(0) = 0$, it starts from rest. If the constant G is chosen so that $p = 0$ for some value of yj^{-1}, then p remains zero for this value of yj^{-1}, which therefore represents a "free" surface of the gas. Thus the flow represents the free expansion of an isentropic gas into vacuum. The equation of the free surface is $y = \frac{y_o}{j(0)}\, j(t)$, where y_o represents the location of the free surface at $t = 0$. In a similar way (24)-(26) may represent the free expansion into vacuum of a gas with initially variable entropy, provided $g_o = 0$ and $F(yj^{-1})$ vanishes for one value of its argument. For certain choices of $j(t)$, this solution can also represent the collapse or expansion of a hole, or the initial contraction of a sphere of gas toward the origin due to an initial inward motion, followed by momentary rest and subsequent expansion. When g_o is not zero, similar motions of a liquid are described.

VIII. Other Solutions

In the preceding sections product solutions of (3) were considered. In this section two other kinds of solutions will be examined. First, let us assume that

$$(77) \qquad y = f(\alpha t + \beta h) \quad .$$

Here α and β are constants. We find for the flow quantities

$$(78) \qquad u = y_t = \alpha\dot{f} = \alpha\dot{f}[f^{-1}(y)]$$

$$(79) \qquad \tau = \rho^{-1} = \beta\alpha^{-1}y^{n-1}u$$

$$(80) \qquad p = A\rho^{\gamma} = A(\alpha\beta^{-1}y^{2-n}u^{-1})^{\gamma} \quad .$$

We see that the flow quantities are independent of t, or in other words, that a solution of the form (77) yields steady flows.

Inserting (77) into (8) yields

$$(81) \qquad \alpha^2\ddot{f} = \gamma A(\beta f^{n-1}\dot{f})^{-\gamma-1}(\beta f^{n-1}\dot{f})^{\cdot}\beta f^{n-1} + A_h(\beta f^{n-1}\dot{f})^{-\gamma}f^{n-1} \quad .$$

Since f is a function of $\alpha t + \beta h$ and t doesn't appear in (81), h cannot appear either. Thus A is constant and $A_h = 0$. Now if x denotes the argument of f, and if we consider the inverse function x(f), we obtain from (81) the following equation for $q(f) \equiv \dot{x}(f)$

$$(82) \qquad -\alpha^2\dot{q}q^{-3} = \gamma A(\beta f^{n-1}q^{-1})^{-\gamma-1}(-\beta f^{n-1}\dot{q}q^{-3} + \beta(n-1)f^{n-2}q^{-2})\beta f^{n-1} .$$

This equation is exact and yields upon integration

$$(83) \qquad bf^{(1-\gamma)(n-1)}q^{\gamma-1} + \frac{1}{2}q^{-2} = D \ , \quad b = \frac{\gamma A\beta^{1-\gamma}}{\alpha^2(\gamma-1)} \quad , \quad \gamma \neq 1$$

$$(84) \qquad \frac{\alpha^2}{2}q^{-2} + A\log q = (n-1)Af + D \qquad , \quad \gamma = 1 \quad .$$

Here D is a constant.

From (78) and the facts that $\dot{f} = q^{-1}$, $f = y$, we have

(85) $$u = \frac{c}{q(y)} \quad .$$

Thus the steady solutions are obtained from the algebraic equation (83) or (84).

A more interesting type of solution is given by

(86) $$y = t^a f(K(h)t) \quad .$$

Here a is a constant, and f, K are functions to be determined. These solutions are like the type considered by Guderly, Taylor, von Neumann and Calkin and Courant and Friedrichs.

Inserting (86) into (8) yields

(87) $$(a-1)at^{a-2}f + 2aKt^{a-1}\dot{f} + K^2t^a\ddot{f}$$

$$= \gamma A\left[t^{an+1}f^{n-1}\dot{K}\dot{f}\right]^{-\gamma-1}t^{a(n-1)}f^{n-1}t^{an+1}$$

$$\left[f^{n-1}\dot{f}\ddot{K} + f^{n-1}\dot{K}^2t\ddot{f} + (n-1)\dot{K}^2\dot{f}^2tf^{n-2}\right]$$

$$+ A_h t^{a(n-1)}f^{n-1}\left[t^{an+1}f^{n-1}\dot{K}\dot{f}\right]^{-\gamma} \quad .$$

In order that h occur in this equation only in the combination $x = K(h)t$, we find the following expressions for K(h) and A(h)

(88) $$K = \left(Bh + C\right)^{\frac{1}{1-\varepsilon}} , \qquad A = D\dot{K}^{\gamma-1}K^{c+1} \quad .$$

Here $c = an(1-\gamma)-\gamma-2a+2$, D is a constant and A is constant if $\varepsilon(\gamma-1)+c+1 = 0$.

Under these conditions, (87) becomes

(89) $$(a-1)af + 2ax\dot{f} + x^2\ddot{f} = x^c f^{(n-1)(1-\gamma)}\dot{f}^{-\gamma}D$$

$$\left[\gamma\left(\varepsilon + \frac{x\ddot{f}}{\dot{f}} + \frac{(n-1)x\dot{f}}{f}\right) + \varepsilon(\gamma-1) + c + 1\right] \quad .$$

To reduce (89) to a first order equation, we follow von Neumann and Calkin, by letting $\delta = (a+\gamma)[2+n(\gamma-1)]^{-1}$ and

$$(90)\qquad x = e^{s} \quad , \quad f = e^{\delta s}F \quad .$$

Then (89) becomes

$$(91)\qquad (a-1)aF + 2a\left(\frac{d}{ds}+\delta\right)F + \left(\frac{d}{ds}+\delta-1\right)\left(\frac{d}{ds}+\delta\right)F$$

$$= F^{(n-1)(1-\gamma)}\left[\left(\frac{d}{ds}+\delta\right)F\right]^{-\gamma} D\left[\gamma\varepsilon + \frac{\gamma\left(\frac{d}{ds}+\delta-1\right)\left(\frac{d}{ds}+\delta\right)F}{\left(\frac{d}{ds}+\delta\right)F}\right.$$

$$\left. + \frac{\gamma(n-1)\left(\frac{d}{ds}+\delta\right)F}{F} + \varepsilon\gamma - \varepsilon + a + 1\right] \quad .$$

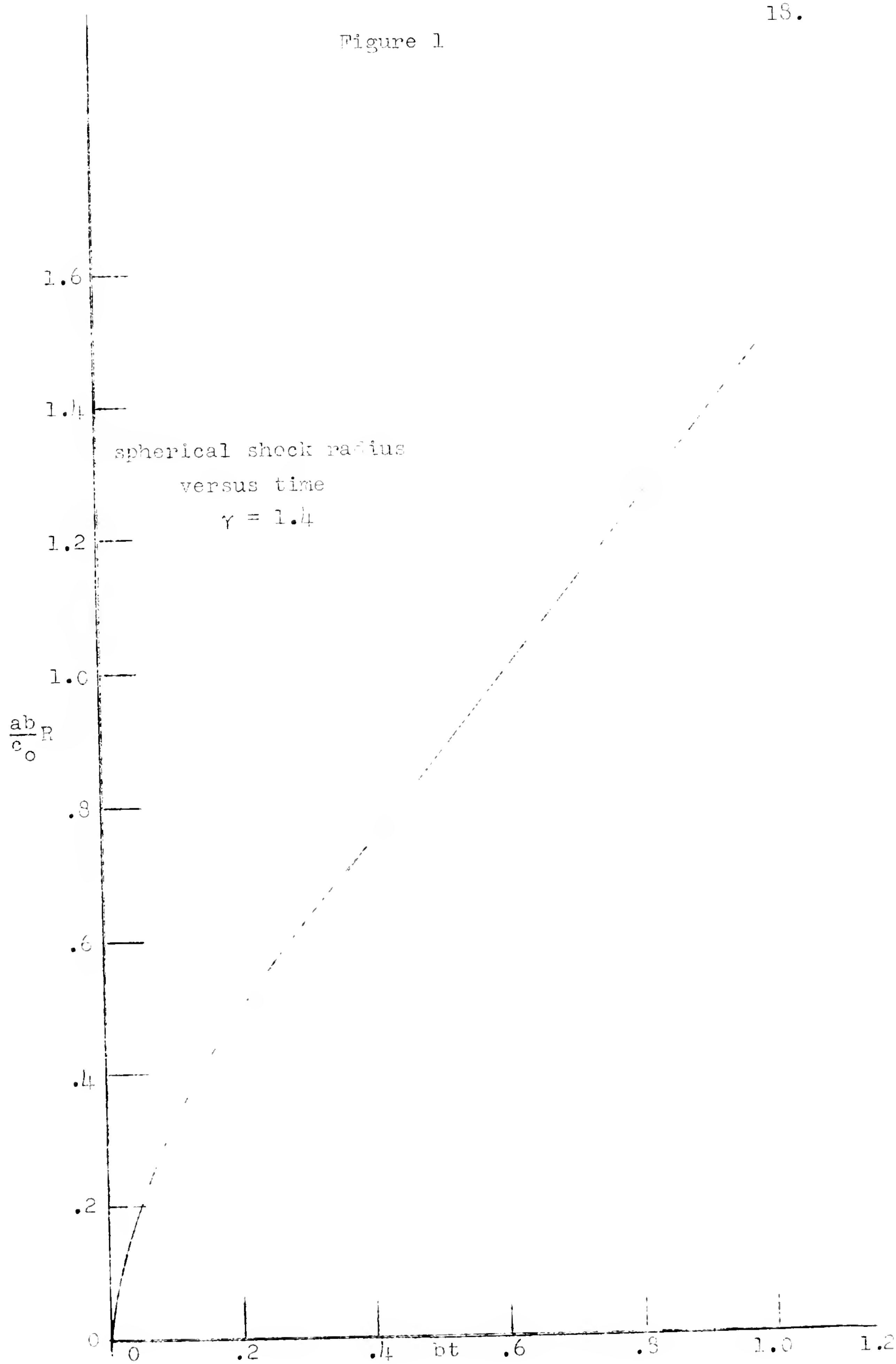
If we now let $q = \left(\frac{d}{ds}+\delta\right)F$ we obtain a first order equation for q in terms of F.

$$(92)\qquad \left[1 - \gamma D F^{(n-1)(1-\gamma)} q^{-\gamma-1}\right](q-\delta F)\dot{q}$$

$$= F^{(n-1)(1-\gamma)}q^{-\gamma}D[\gamma\varepsilon+\gamma(n-1)qF^{-1}+\varepsilon\gamma+a+1-\varepsilon]-(a-1)aF-2aq \quad .$$

Special cases of this equation have been studied by von Neumann and Calkin.

Figure 1


spherical shock radius
versus time
$\gamma = 1.4$
$\frac{ab}{c_o}R$
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
0
.2
.4
bt
.6
.8
1.0
1.2

Date Due

MAR 21 '66			
[illegible] 25 '58			
[illegible]			
[illegible] 8 1992			

IMM-NYU 185

Keller, J. B.

Spherical, cylindrical...